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Polar representations of compact groups and convex hulls of their orbits

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ABSTRACT

The paper contains a characterization of compact groups $G \subseteq \mathrm{GL}(\mathfrak{v})$, where $\mathfrak{v}$ is a finite-dimensional real vector space, which have the following property SP: the family of convex hulls of G-orbits is a semigroup with respect to the Minkowski addition. If G is finite, then SP holds if and only if G is a Coxeter group; if G is connected then SP is equivalent to the property to be polar. In general, G satisfies SP if and only if it is polar and its Weyl group is a Coxeter group.

1. Introduction

A representation of a compact Lie group G in a finite-dimensional Euclidean vector space $\mathfrak{v}$ is called *polar* if there exists a linear subspace $\mathfrak{a} \subset \mathfrak{v}$ (which is said to be a *Cartan subspace*) such that

(A) each orbit $O_v = Gv$, where $v \in \mathfrak{v}$, meets $\mathfrak{a}$;
(B) for any $u \in \mathfrak{a}$, the tangent space $\mathfrak{t}_u = T_u O_u$ is orthogonal to $\mathfrak{a}$.

It follows that the set $O_v \cap \mathfrak{a}$ is finite and $\mathfrak{a} = \mathfrak{t}_u^\perp$ for generic $u \in \mathfrak{a}$. Polar representations of compact Lie groups were defined and described in [4]. An example is the adjoint representation Ad of G in its Lie algebra $\mathfrak{g}$, with any Cartan subalgebra as a Cartan subspace. A more general example is the isotropy representation of a Riemannian symmetric space (an *s-representation*). Let $M = H/K$ be such a space, $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{v}$ be the Cartan decomposition, where $\mathfrak{h}, \mathfrak{k}$ are Lie algebras of H, K, respectively, and the space $\mathfrak{v}$ is $\mathrm{Ad}(K)$-invariant. The representation Ad of K in $\mathfrak{v}$ is polar since any maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{v}$ satisfies (A) and (B). By a result of Dadok [4], a representation of G is polar if and only if it is *orbit equivalent* to some s-representation: there exist H, K as above and an embedding $G \mapsto K$ such that G is transitive on each K-orbit in $\mathfrak{v}$. In the paper [4], the proof involves a case-by-case check. A conceptual proof (in the case that the cohomogeneity is bigger than two) was given by Eschenburg and Heintze in the papers [6,7]. An alternative approach was realized by Kollross who proved in the paper [9] that the irreducible polar representations are characterized by the following two simple geometric properties: they are orbit maximal in the unit sphere S (i.e., any action with larger orbits is transitive on S) and have low codimension (this means that the dimension of a generic orbit is greater than or equal to $\binom{k+1}{2}$, where k is its codimension).

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The paper [5] by Dadok and Kac contains a classification of polar representations of complex reductive Lie groups (there is a natural extension of the definition to this case). The family of polar representations is not wide, see the papers [4–7] for detailed information. The survey [14] contains a comparison of the property to be polar with other "good" properties of representations, for example, to have a free algebra of invariant polynomials.

In this article, we characterize the polar representations of compact Lie groups by a semigroup property of their orbits. Let G be a compact subgroup of $\mathrm{GL}(\mathfrak{v})$; we say that G is polar if its identical representation is polar. For $A, B \subseteq \mathfrak{v}$,

$$A + B = \{a + b\colon\ a \in A,\ b \in B\}$$

is the Minkowski sum of A and B. Let $\widehat{X}$ denote the convex hull of a set $X \subseteq \mathfrak{v}$. Here is the semigroup property of G mentioned above:

SP: the family $\{\widehat{O}_v\}_{v\in\mathfrak{v}}$ of convex hulls of orbits is a semigroup with respect to the Minkowski addition.

The group

$$W = \{g \in G\colon\ g\mathfrak{a} = \mathfrak{a}\}|_{\mathfrak{a}} \tag{1}$$

is said to be the *Weyl group* of G. Due to (B), it is finite. In the statement of the following theorem, which is the main result of the paper, Coxeter groups are treated as finite linear groups generated by reflections in hyperplanes.

Theorem 1. *A compact linear group satisfies* SP *if and only if it is polar and its Weyl group is a Coxeter group.*

In the definition of a polar representation, it is not assumed that G is connected. However, this property depends only on the identity component G^e of G; in particular, all finite linear groups are polar by definition. By the theorem, a finite linear group satisfies SP if and only if it is a Coxeter group, including non-crystallographic ones. This is proved in Theorem 2, which also contains two other geometric criteria for SP (hence for a finite linear group to be Coxeter).

Let G be connected. If G is polar, then generic orbits are isoparametric submanifolds in the ambient Euclidean space (a submanifold of the Euclidean space is called *isoparametric* if its normal bundle is flat and principal curvatures are constant for any parallel normal vector field). For isoparametric submanifolds of codimension greater than 2 the converse is true (i.e., they can be realized as principal orbits of polar groups, see [13]; in codimension 2, there are nonhomogeneous examples). Any compact connected isoparametric submanifold is naturally associated with a Coxeter group (see, for example, [10, Section 6.3]). If the submanifold is an orbit of a polar group, then this Coxeter group coincides with the Weyl group. Thus, if G is connected, then SP holds if and only if G is polar. The proof of these facts uses the Morse theory; it would be interesting to know if there exists a direct elementary proof of SP for connected polar groups.

Let H be a subgroup of $\mathrm{GL}(\mathfrak{v})$ and $\mathfrak{C}(\mathfrak{v}, H)$ be the family of all H-invariant convex sets in $\mathfrak{v}$. Clearly, $\mathfrak{C}(\mathfrak{v}, H)$ with the Minkowski addition is a semigroup. The following proposition is an essential step in the proof of the theorem.

Proposition 1. *If G is polar, then the mapping $A \to A \cap \mathfrak{a}$, where $\mathfrak{a}$ is a Cartan subspace, is a semigroup isomorphism between $\mathfrak{C}(\mathfrak{v}, G)$ and $\mathfrak{C}(\mathfrak{a}, W)$.*

The semigroups of sets in topological groups were considered in papers [11,12,2,1]. In $\mathbb{R}^n$, one parameter semigroups are of the form $\{tQ\}_{t\geqslant 0}$, where Q is a convex set. In particular, the family of closed balls for any norm is a semigroup. For a left invariant Riemannian metric in a Lie group, the family of closed balls centered at the identity is also a semigroup of sets. This property holds for all left invariant inner metrics, moreover, it characterizes them (see [2]). Semigroups may be parameterized by more general objects than the numbers. In terms of [2], this defines a geometry on a group. The orbits of an s-representation are parameterized by points of a closed convex simplicial cone C (the Weyl chamber of the restricted root system). Thus, the semigroup of their convex hulls can be treated as a vector valued norm (in general, non-symmetric) on $\mathfrak{v}$ with values in C.

The exposition is self-contained and elementary. Some of preliminary results were published in [8]. We refer to [3] and [14] for general facts on actions of groups.

Throughout the paper, we keep the notation above. Furthermore, let f be a real function on a set X. The set of all $x \in X$ such that $f(x) = \max_{y\in X} f(y)$ is called the *peak set for f on X*. A *peak point* is a point which is a peak set; in this case, we say that *f has a peak on X*. $\mathrm{Int}_{\mathfrak{l}}(X)$ is the interior of a set $X \subseteq \mathfrak{l}$ in $\mathfrak{l} \subseteq \mathfrak{v}$ (we drop the index if $\mathfrak{l} = \mathfrak{v}$). The space $\mathfrak{v}$ is equipped with a G-invariant inner product $\langle\ ,\ \rangle$. For $v, u \in \mathfrak{v}$ and compact $X \subseteq \mathfrak{v}$, set

$$\mathfrak{a}_v = \mathfrak{t}_v^{\perp} \subseteq \mathfrak{v},$$

$$\lambda_v(u) = \langle v, u\rangle,$$

$$\mu_v(X) = H_X(v) = \max_{x\in X} \lambda_v(x),$$

$$P_v(X) = \bigl\{x \in X\colon\ \lambda_v(x) = \mu_v(X)\bigr\}.$$

H_X is the support function for X. Note that u is a critical point for λ_v on O_u if and only if $v \in \mathfrak{a}_u$, equivalently, if and only if $u \in \mathfrak{a}_v$ ($\mathfrak{g}u \perp v$ is the same as $u \perp \mathfrak{g}v$). Since $P_v(O_u)$ is the peak set for λ_v on O_u, we get

$$P_v(O_u) \subseteq \mathfrak{a}_v. \tag{2}$$

For all compact $X, Y \subseteq \mathfrak{v}$ and $v \in \mathfrak{v}$, we obviously have

$$\mu_v(X+Y) = \mu_v(X) + \mu_v(Y), \tag{3}$$

$$P_v(X+Y) = P_v(X) + P_v(Y). \tag{4}$$

The stable subgroup of $v \in \mathfrak{v}$ and its Lie algebra are denoted by G_v, $\mathfrak{g}_v$, respectively. A point $v \in \mathfrak{v}$ is said to be *regular* if G_v is minimal: $G_u \subseteq G_v$ implies $G_u = G_v$, where $u \in \mathfrak{v}$. For $X \subseteq \mathfrak{v}$, X^{reg} is the set of all regular points in X; $\mathrm{span}(X)$ is the linear span of X. The algebra of linear operators $\mathfrak{v} \to \mathfrak{v}$ is denoted by $\mathrm{L}(\mathfrak{v})$, e is the unit of G, $\pi \in \mathrm{L}(\mathfrak{v})$ is the orthogonal projection onto the Cartan subspace $\mathfrak{a}$, and $\mathbb{R}^+ = [0, \infty)$.

2. Preparatory material

In this section, G is not assumed polar unless this is stated explicitly. Obviously, the sum of convex sets is convex and the sum of G-invariant sets is G-invariant. Hence the inclusion

$$\widehat{O}_u + \widehat{O}_v \supseteq \widehat{O}_{u+v}$$

holds for all $u, v \in \mathfrak{v}$. The family $\{\widehat{O}_v\}_{v\in\mathfrak{v}}$ is a semigroup if and only if the equality holds for some u, v in every pair of orbits.

The tangent space $\mathfrak{t}_v = T_v O_v$ may be identified with the quotient $\mathfrak{g}/\mathfrak{g}_v$ or with the complementary subspace:

$$\mathfrak{t}_v = \mathfrak{g}v \cong \mathfrak{g}/\mathfrak{g}_v = \mathfrak{g}_v^{\perp} \subseteq \mathfrak{g},$$

where $\perp$ relates to some invariant inner product in $\mathfrak{g}$; then $\mathfrak{g}_v^{\perp}$ is $\mathrm{ad}(\mathfrak{g}_v)$-invariant.

Lemma 1. *For any $u \in \mathfrak{v}$, there exists a neighborhood U of u in $\mathfrak{a}_u$ such that λ_v has a peak on O_u at u for all $v \in U$.*

Proof. Clearly, u is a peak point for λ_u on O_u and $d_u^2\lambda_u$ is negative definite on $\mathfrak{t}_u$. If $v - u$ is small, then the latter is also true for $d_u^2\lambda_v$; if $v \in \mathfrak{a}_u$, then u is a critical point for λ_v. Taken together, the two properties imply that λ_v has a strict local maximum on O_u at u, which is global if $v \in \mathfrak{a}_u$ is sufficiently close to u. □

Let C_v^* be the closed convex cone hull of the shifted orbit O_{-v} and C_v be the dual cone to it:

$$C_v = \{u \in \mathfrak{v}\colon\ \langle u, v - gv\rangle \geqslant 0 \text{ for all } g \in G\},$$

$$C_v^* = \mathrm{clos}\big(\mathbb{R}^+(v - \widehat{O}_v)\big). \tag{5}$$

Clearly, $v \in C_v$. It follows from (5) that

$$u \in C_v \iff v \in P_u(O_v). \tag{6}$$

Corollary 1. *C_v is a closed convex generating cone in $\mathfrak{a}_v$.*

Proof. Clearly, C_v is convex and closed. For all $u, v \in \mathfrak{v}$ we have

$$\mu_u(O_v) = \mu_v(O_u), \tag{7}$$

$$u \in C_v \iff v \in C_u, \tag{8}$$

$$u \in P_v(O_u) \iff v \in P_u(O_v). \tag{9}$$

Indeed, (7) is true since $\max_{g\in G}\langle u, gv\rangle = \max_{g\in G}\langle v, g^{-1}u\rangle$, (8) and (9) follow from (6), (7), and the evident equality $\lambda_u(v) = \lambda_v(u)$. By (6), (9), and (2), $C_v \subseteq \mathfrak{a}_v$, moreover, $\mathrm{Int}_{\mathfrak{a}_v}(C_v) \neq \emptyset$ due to Lemma 1. □

According to Corollary 1, $\mathfrak{t}_v = C_v^{\perp}$. Hence,

$$C_v^* = \big(C_v^* \cap \mathfrak{a}_v\big) + \mathfrak{t}_v. \tag{10}$$

Note that C_v is pointed (and C_v^* is generating) if and only if $\mathrm{Int}(\widehat{O}_v) \neq \emptyset$. By (6) and (9),

$$P_v(O_u) = C_v \cap O_u. \tag{11}$$

Corollary 2. *Each G-orbit in $\mathfrak{v}$ meets C_v.*

Proof. Since G is compact, we have $\lambda_v(x) = \mu_v(O_u)$ for some $x \in O_u$. $\square$

Note that each orbit of any connected component of G also meets $\mathfrak{a}_v$ since there are critical points of λ_v in it but the intersection need not have a common point with C_v.

It follows from (11) that

$$O_v \cap C_v = \{v\}. \tag{12}$$

Lemma 2. *Let $u \in C_v$. If λ_v has a peak on O_u at u, then $G_v \subseteq G_u$.*

Proof. If $g \in G_v$, then $gu \in C_v$ by (8) since $v = gv \in gC_u = C_{gu}$. By (11), $gu \in P_v(O_u)$. Since u is a peak point for λ_v, we get $gu = u$. $\square$

Lemma 3. *Let $u \in O_v$ and $u \neq v$. Set $w = u - v$. Then $(C_u \cap C_v) \perp w$. Furthermore, $w \notin \mathfrak{a}_v^\perp$ and the set $C_u \cap C_v$ is contained in the hyperplane $\mathfrak{a}_v \cap w^\perp$ in $\mathfrak{a}_v$, which is proper. The hyperplane $w^\perp$ separates C_u and C_v, moreover, u and v are strictly separated.*

Proof. If $x \in C_u \cap C_v$, then $\lambda_x(u) = \lambda_x(v) = \mu_x(O_v)$ by (6). Hence $\langle x, w\rangle = 0$. Further, $w \perp \mathfrak{a}_v$ implies $|u|^2 = |v|^2 + |w|^2 > |v|^2$ contradictory to $u \in O_v$. For arbitrary $x \in C_u$ and $y \in C_v$,

$$\langle w, x\rangle = \lambda_x(u) - \lambda_x(v) \geqslant 0 \geqslant \lambda_y(u) - \lambda_y(v) = \langle w, y\rangle.$$

The inequalities hold due to (6). If $x = u$ or $y = v$, then at least one of the inequalities is strict; if $x = y \in C_u \cap C_v$, then we have equalities. $\square$

Note that $w^\perp$ is the equidistant hyperplane for u and v since $|u| = |v|$.

Corollary 3. *The following assertions hold*:

(1) *if $u \in O_v$ and $u \neq v$, then $\mathrm{Int}_{\mathfrak{a}_v}(C_v) \cap C_u = \emptyset$;*
(2) *if $x \in \mathrm{Int}_{\mathfrak{a}_v}(C_v)$, then $C_v \cap O_x = G_v x$.*

Proof. By Lemma 3, the hyperplane $w^\perp$ strictly separates $\mathrm{Int}_{\mathfrak{a}_v}(C_v)$ and C_u. This implies (1). If $g \in G$, $x \in \mathrm{Int}_{\mathfrak{a}_v} C_v$, and $gx \in C_v$, then

$$x \in \mathrm{Int}_{\mathfrak{a}_v}(C_v) \cap C_{g^{-1}v}.$$

By (1), $gv = v$. Conversely, if $gv = v$, then $gC_v = C_v$, hence $gx \in C_v$. This proves (2). $\square$

We omit the proof of the following lemma, which is standard.

Lemma 4. *For any $u \in \mathfrak{v}^{\mathrm{reg}}$ and its neighborhood V in $\mathfrak{v}$, there exists a neighborhood U of u in $\mathfrak{v}$ with the following property*: *if $v \in U$, then $V \cap O_v$ contains a unique critical point of λ_u on O_v, which is also a peak point for λ_u on O_v.*

Lemma 5. *A vector $v \in \mathfrak{v}$ is regular if and only if $G_v \subseteq G_u$ for all $u \in \mathfrak{a}_v$.*

Proof. According to Lemma 2, for all u in the set U of Lemma 1 the reverse inclusion $G_u \subseteq G_v$ holds; if v is regular, then $G_u = G_v$. Since U is open in $\mathfrak{a}_v$, $G_v \subseteq G_u$ for all $u \in \mathfrak{a}_v$. Conversely, let $G_v \subseteq G_u$ for all $u \in \mathfrak{a}_v$. It follows from Corollary 2 that $\mathfrak{a}_v^{\mathrm{reg}} \neq \emptyset$. If $u \in \mathfrak{a}_v^{\mathrm{reg}}$, then $G_u = G_v$ since G_u is minimal. Thus, v is regular. $\square$

We conclude this section with a proposition which combines the facts on polar groups that we need in the sequel.

Proposition 2. *Let G be polar, $\mathfrak{a}$ be a Cartan subspace, and W be the Weyl group. Then $\mathfrak{a}^{\mathrm{reg}}$ is open and dense in $\mathfrak{a}$ and $\mathfrak{a} = \mathfrak{a}_v$ for any $v \in \mathfrak{a}^{\mathrm{reg}}$. Furthermore,*

(i) *if $v \in \mathfrak{a}^{\mathrm{reg}}$, $g \in G$, and $gv \in \mathfrak{a}$, then $g\mathfrak{a} = \mathfrak{a}$;*
(ii) *if $v \in \mathfrak{v}^{\mathrm{reg}}$, then G_v is a normal subgroup of finite index in the group*

$$G^v = \{g \in G\colon\ g\mathfrak{a}_v = \mathfrak{a}_v\} = \{g \in G\colon\ gv \in \mathfrak{a}_v\}$$

and $G^v/G_v \cong W$;

(iii) *for all* $a \in \mathfrak{a}$, $Wa = O_a \cap \mathfrak{a}$;
(iv) *for any* $a \in \mathfrak{a}$, $\pi \widehat{O}_a = \widehat{Wa}$.

Proof. It follows from (A) that $\mathfrak{a}^{\mathrm{reg}} \neq \emptyset$. Let $v \in \mathfrak{a}^{\mathrm{reg}}$. By (B), $\mathfrak{t}_v \perp \mathfrak{a}$. Hence $\mathfrak{a}_v \supseteq \mathfrak{a}$. If $\operatorname{codim} \mathfrak{t}_v > \dim \mathfrak{a}$, then $\dim(\mathfrak{v}/G) > \dim \mathfrak{a}$, contradictory to (A) and (B). Thus, $\mathfrak{a}_v = \mathfrak{a}$. It is well known that $\mathfrak{v}^{\mathrm{reg}}$ is open and dense in $\mathfrak{v}$. Hence, $\mathfrak{a}^{\mathrm{reg}}$ is open in $\mathfrak{a}$; it follows from (A) and (B) that $\mathfrak{a}^{\mathrm{reg}}$ is dense in $\mathfrak{a}$.

If $v \in \mathfrak{a}^{\mathrm{reg}}$ and $gv \in \mathfrak{a}$, then $gv \in \mathfrak{a}^{\mathrm{reg}}$. Hence $\mathfrak{a} = \mathfrak{a}_{gv}$. Since $g\mathfrak{a}_v = \mathfrak{a}_{gv}$ and $\mathfrak{a}_v = \mathfrak{a}$, this implies (i).

In (ii), we may assume $v \in \mathfrak{a}^{\mathrm{reg}}$; then $\mathfrak{a}_v = \mathfrak{a}$. By Lemma 5, $G_u = G_v$ for all $u \in \mathfrak{a}^{\mathrm{reg}}$. Therefore, $G_{gv} = G_v$ if $g \in G^v$, G_v is normal in G^v, and we have $G^v/G_v \cong G^v|_{\mathfrak{a}} = W$. By (i), $O_v \cap \mathfrak{a} = G^v v$; thus $Wv = O_v \cap \mathfrak{a}$. It follows from (B) that $O_v \cap \mathfrak{a}$ and W are finite.

For $a \in \mathfrak{a}^{\mathrm{reg}}$, (iii) was proved above; for all $a \in \mathfrak{a}$, (iii) is true since $\mathfrak{a}^{\mathrm{reg}}$ is dense in $\mathfrak{a}$.

In (iv), the inclusion $\pi \widehat{O}_a \supseteq \widehat{Wa}$ is obvious. By (iii), we have to prove that $\pi \widehat{O}_a \subseteq \widehat{O_a \cap \mathfrak{a}}$. Otherwise, there exist $u \in O_a$ and $b \in \mathfrak{a}^{\mathrm{reg}}$ such that

$$\lambda_b(u) > \max\{\lambda_b(x)\colon x \in O_a \cap \mathfrak{a}\}.$$

Then $P_b(O_u) \cap \mathfrak{a} = \emptyset$ but this contradicts to (2) since $\mathfrak{a} = \mathfrak{a}_b$. $\square$

3. SP for finite linear groups

If G is finite, then $\mathfrak{a}_v = \mathfrak{v}$ for all $v \in \mathfrak{v}$ and $\mathfrak{v}^{\mathrm{reg}}$ consists of $v \in \mathfrak{v}$ such that $G_v = \{e\}$. The following lemma is a specification of Lemma 3 to this case.

The Dirichlet–Voronoi partition of $\mathfrak{v}$ can be defined for any discrete subset $L \subset \mathfrak{v}$: each $x \in L$ corresponds to a domain D_x which consists of points $v \in \mathfrak{v}$ such that $|v - x|$ equals to the distance between v and L. These domains form the partition. If L is a generic orbit of a discrete group acting in $\mathfrak{v}$ properly by isometries, then D_x is a fundamental domain for it.

Lemma 6. *For any $v \in \mathfrak{v}$, the family of cones $\{C_{gv}\}_{g\in G}$ defines the Dirichlet–Voronoi partition of $\mathfrak{v}$ for the orbit O_v:*

$$C_{hv} = \Bigl\{u \in \mathfrak{v}\colon |u - hv| = \min_{g\in G} |u - ghv|\Bigr\}. \tag{13}$$

Proof. Let $v \in \mathfrak{v}$. By definition, $-C_v^*$ is the tangent cone to the convex polytope $\widehat{O}_v$ at the vertex v. The dual cone C_v is uniquely determined by the following properties: it contains v and is bounded by hyperplanes which are orthogonal to extreme rays of C_v^*. Let $\mathbb{R}^+(v - gv)$, $g \in G$, be such a ray. Then the equidistant hyperplane H for v and gv defines a face of C_v (note that $|v| = |gv|$ implies $0 \in H$). By (12), $O_v \cap C_v = \{v\}$. Hence, $u \in C_v$ if and only if $|u - v| \leqslant |x - v|$ for all $x \neq v$ in O_v. This proves (13) for $h = e$ that is evidently sufficient. $\square$

Corollary 4. *If $v \in \mathfrak{v}^{\mathrm{reg}}$, then the action of G on the family of cones C_{gv}, $g \in G$, is simply transitive.*

Theorem 2. *Let G be finite. Then* SP *is equivalent to each of the following properties*:

(i) *for any $v \in \mathfrak{v}^{\mathrm{reg}}$, the functional λ_v has a peak on each G-orbit in $\mathfrak{v}$;*
(ii) *G is a Coxeter group;*
(iii) *if $v \in \mathfrak{v}^{\mathrm{reg}}$, then $C_u = C_v$ for any u from some neighborhood of v.*

Proof. SP $\Rightarrow$ (i). Let $v \in \mathfrak{v}^{\mathrm{reg}}$, $u, w \in \mathfrak{v}$, and $\widehat{O}_v + \widehat{O}_u = \widehat{O}_w$. Set $P = P_v(O_u)$. Due to Corollary 2, we may assume $u, w \in C_v$; then $u \in P$ by (11). Clearly, $P_v(\widehat{O}_v) = P_v(O_v) = \{v\}$ and $P_v(\widehat{O}_u) = \widehat{P}$. By (4), we have

$$P_v(\widehat{O}_w) = v + \widehat{P}.$$

Furthermore, $v + \widehat{P} \subset \operatorname{Int}(C_v)$ since $v \in \operatorname{Int}(C_v)$, $\widehat{P} \subseteq C_v$, and $\widehat{P}$ is compact. Let E be the set of extreme points of $v + \widehat{P}$. Then $E \subseteq v + P \subset \operatorname{Int}(C_v)$. On the other hand, $E \subseteq O_w$ since extreme points of $P_v(\widehat{O}_w)$ are extreme points of $\widehat{O}_w$. By Corollary 3, $O_w \cap C_v = G_v w$. The stable subgroup G_v is trivial since $v \in \mathfrak{v}^{\mathrm{reg}}$. Therefore, $v + P = \{w\}$ and $P = \{u\}$.

(i) $\Rightarrow$ (ii). Let $v \in \mathfrak{v}^{\mathrm{reg}}$, $u \in O_v$, $u \neq v$. Due to Lemma 6, we may assume that C_v and C_u have a common wall

$$\mathcal{W} = C_v \cap C_u$$

which is contained in an equidistant hyperplane H such that $\operatorname{Int}_H \mathcal{W} \neq \emptyset$. Let $w \in \operatorname{Int}_H(\mathcal{W})$. Then $u, v \in P_w(O_v)$. It follows from (i) that w is not regular. Hence, $G_w \neq \{e\}$. Let $g \in G_w \setminus \{e\}$. Then $gC_v \neq C_v$ by Corollary 4. Obviously, w has a G_w-invariant neighborhood U such that $U \subseteq C_v \cup C_u$; this implies $gC_v = C_u$. Similar arguments show that $gC_u = C_v$. Therefore, $g^2 C_v = C_v$. By Corollary 4, $g^2 = e$ and the condition $gC_v = C_u$ uniquely determines g. The latter means that g is independent of the choice of $w \in \operatorname{Int}_H(\mathcal{W})$. Hence g is identical on some open subset of H. Consequently, g is a nontrivial involution that fixes points of H. Thus, g is the reflection in H. Every pair of cones in the family $\{C_{gv}\}_{g\in G}$ can be joined by a chain of these cones in such a way that consecutive ones have a common wall; since G acts freely on $\{C_{gv}\}_{g\in G}$ by Corollary 4, G is generated by reflections.

(ii) ⇒ (iii), (ii) ⇒ SP. Let G be a Coxeter group and C be a Weyl chamber. Then C is a simplicial cone; let $\varpi_1,\dots,\varpi_n$ be a base in $\mathfrak{v}$ such that $C=\sum_{k=1}^n \mathbb{R}^+\varpi_k$ and let $\alpha_1,\dots,\alpha_n$ be the dual base, which generates the dual cone C^*. The group G_{ϖ_k} is generated by reflections in those walls of C that contain ϖ_k (they correspond to α_j with $j\neq k$). Hence, $\widehat{G_{\varpi_k}v}\subset\widehat{O}_v$; moreover, $\widehat{G_{\varpi_k}v}$ is a face of $\widehat{O}_v$ which is orthogonal to ϖ_k. This proves inclusions $\widehat{O}_v\cap C\supseteq(v-C^*)\cap C$ and $C_v\supseteq C$. On the other hand, the set $\bigcap_{g\in G} g(v-C^*)$ is convex and contains v. Therefore,

$$\widehat{O}_v=\bigcap_{g\in G} g\bigl(v-C^*\bigr)=\bigcup_{g\in G} g\bigl(\bigl(v-C^*\bigr)\cap C\bigr) \tag{14}$$

and $C_v=C$. This proves (iii). Further, SP evidently holds for the families $\{(v-C^*)\cap C\}_{v\in C}$ and $\{g(v-C^*)\}_{v\in C}$ for any $g\in G$, hence for $\widehat{O}_v$.

(iii) ⇒ (ii). Let $C_v=C_u=C$ for u in some open set U. Then $C_v^*=C_u^*=C^*$, where C^* is the dual cone to C. Since $v\in\mathfrak{v}^{\mathrm{reg}}$, we have $G_v=\{e\}$. Hence, we may assume that $gu\notin hU$ if $g\neq h$, taking a smaller U if necessary. This implies that each extreme ray of C^* is of the form $\mathbb{R}^+(u-gu)$, where $g\in G$ does not depend on $u\in U$. Therefore, the linear operator $\mathbb{1}-g$ (where $\mathbb{1}$ is the identical transformation) maps U into some one-dimensional subspace and is nontrivial. Hence it has rank 1. Since g is orthogonal, it is a reflection. It remains to note that the action of G on O_v is simply transitive and that every two vertices of a convex polytope can be joined by a chain of one-dimensional edges. □

4. Proof of the main result

For a polar G, $\mathfrak{a}$ is a Cartan subspace, π is the orthogonal projection onto $\mathfrak{a}$, and W is the Weyl group; G^e denotes the identity component of G.

Lemma 7. *Let G be polar. Then*

(1) *for any G-invariant convex set $Q\subseteq\mathfrak{v}$, $\pi Q=Q\cap\mathfrak{a}$ and $Q=G(Q\cap\mathfrak{a})$;*
(2) *for every W-invariant convex set $A\subseteq\mathfrak{a}$, GA is convex and $\pi GA=A$.*

Proof. Clearly, $\pi\widehat{O}_v\supseteq\widehat{O}_v\cap\mathfrak{a}\supseteq\widehat{O_v\cap\mathfrak{a}}$. Together with Proposition 2(iii) and (iv), this implies $\pi\widehat{O}_v=\widehat{O}_v\cap\mathfrak{a}$ for all $v\in\mathfrak{v}$. Since $Q=\bigcup_{v\in Q}\widehat{O}_v$, this proves the first equality in (1); the second follows from (A).

If $a\in A$, then $\widehat{Wa}\subseteq A$ and we get $\pi GA=A$ since $\pi\widehat{O}_a=\widehat{Wa}$ for all $a\in\mathfrak{a}$ by Proposition 2(iv). Thus, $GA\subseteq\pi^{-1}(A)$. Clearly, $\mathfrak{a}\cap\pi^{-1}(A)=A$ and the set $\pi^{-1}(A)$ is convex. For any $g\in G$, the same is true for the set gA, the space $g\mathfrak{a}$, and the orthogonal projection π_g onto it. Hence $GA=\bigcap_{g\in G}\pi_g^{-1}(gA)$. This proves that GA is convex. Thus, (2) is true. □

Proof of Proposition 1. Clearly, π induces a homomorphism $\mathfrak{C}(\mathfrak{v},G)\to\mathfrak{C}(\mathfrak{a},W)$. Since G is polar, $\pi V=V\cap\mathfrak{a}$ for all $V\in\mathfrak{C}(\mathfrak{v},G)$ by Lemma 7(1). Hence the mapping $\alpha: V\to V\cap\mathfrak{a}$, $V\in\mathfrak{C}(\mathfrak{v},G)$ coincides with π on $\mathfrak{C}(\mathfrak{v},G)$. Thus, α is a homomorphism. It follows from Lemma 7(1), that α is one-to-one. By Lemma 7(2), $GA\in\mathfrak{C}(\mathfrak{v},G)$ for any $A\in\mathfrak{C}(\mathfrak{a},W)$ and $\pi GA=A$. Hence α is surjective. □

Proof of Theorem 1. Let G be polar and W be a Coxeter group. Then the family $\{\widehat{Wa}\}_{a\in\mathfrak{a}}$ is a semigroup due to Theorem 2. By Proposition 2(iv), we have $\widehat{O}_a=\widehat{Wa}$ for all $a\in\mathfrak{a}$. It follows from Proposition 1 and Theorem 2 that $\{\widehat{O}_v\}_{v\in\mathfrak{v}}$ is a semigroup, i.e., G satisfies SP.

Conversely, let SP hold for G. Then, for each pair of G-orbits, there exist vectors u,v in them such that $\widehat{O}_u+\widehat{O}_v=\widehat{O}_{u+v}$. Clearly,

$$\widehat{O}_u+\widehat{O}_v=\widehat{O}_{u+v}\quad\Longrightarrow\quad\mathfrak{t}_u+\mathfrak{t}_v\subseteq\mathfrak{t}_{u+v}. \tag{15}$$

Suppose u regular. We claim that

$$\mathfrak{a}=\mathfrak{a}_u$$

is a Cartan subspace. The condition (A) is obvious. By (15), $\dim\mathfrak{t}_u\leqslant\dim\mathfrak{t}_{u+v}$; since $u\in\mathfrak{a}^{\mathrm{reg}}$, we have $\dim\mathfrak{t}_u=\dim\mathfrak{t}_{u+v}$. Therefore, $\mathfrak{t}_u=\mathfrak{t}_{u+v}=\mathfrak{t}_u+\mathfrak{t}_v$. Moreover, $\mathfrak{t}_v=\mathfrak{t}_u$ if v is regular; then

$$\mathfrak{t}_v=\mathfrak{a}^{\perp}. \tag{16}$$

Thus, it is sufficient to prove that there exists a neighborhood U of u in $\mathfrak{a}$ such that

$$v\in U\quad\Longrightarrow\quad\widehat{O}_u+\widehat{O}_v=\widehat{O}_{u+v} \tag{17}$$

to verify (B). Indeed, we may assume $U \subseteq \mathfrak{a}^{\mathrm{reg}}$. Then (17) implies (16) for all $v \in U$; consequently, $\mathfrak{t}_v \perp \mathfrak{a}$ for all $v \in \mathfrak{a}$.

Let U be such that

$$O_v \cap U = \{v\} \tag{18}$$

for each $v \in U$. Since $u \in \mathfrak{v}^{\mathrm{reg}}$ is a peak point for λ_u on O_u, the function λ_u must have a peak on O_v for v near u by Lemma 4. The peak point $v' \in O_v$ depends on v continuously in some neighborhood of u since u is regular and $d_u^2\lambda_u$ is nondegenerate on $\mathfrak{t}_u$. Furthermore, $v' \in \mathfrak{a} = \mathfrak{a}_u$ since v' is a critical point for λ_u on O_v. Thus, for sufficiently small U, (18) implies that $v' = v$ if $v \in U$. In other words, v in (18) is the peak point of λ_u on O_v for all $v \in U$. Further, we have $\widehat{O}_u + \widehat{O}_v = \widehat{O}_w$ for some $w \in \mathfrak{v}$. Clearly, λ_u has a peak at u on $\widehat{O}_u$ as well as on O_u, and the same is true for λ_u, v, $\widehat{O}_v$, and O_v. Therefore, λ_u has a peak on $\widehat{O}_u + \widehat{O}_v$ at $u + v$. This implies that $u + v$ is an extreme point for λ_u on $\widehat{O}_w$ but the set of extreme points of $\widehat{O}_w$ coincides with O_w since O_w is homogeneous. Thus, $u + v \in O_w$ and we get (17). This proves the claim.

Since G is polar, we may apply Proposition 1. According to it, SP for G implies that the family $\{\widehat{O}_a \cap \mathfrak{a}\}_{a\in\mathfrak{a}}$ is a semigroup. The convex hull $\widehat{Wa}$ is the least convex W-invariant set which contains a, and the same is true for $\widehat{O}_a$ and G. The mapping $Q \to Q \cap \mathfrak{a}$ keeps inclusions and is a bijection by Proposition 1. Hence we have $\widehat{O}_a \cap \mathfrak{a} = \widehat{Wa}$ for all $a \in \mathfrak{a}$. Therefore, $\{\widehat{Wa}\}_{a\in\mathfrak{a}}$ is a semigroup; by Theorem 2, W is a Coxeter group. □

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